First-Order Impulsive Solutions

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Theme

THIS paper gives a mathematically rigorous derivation of first-order corrections to multi-impulse approximations to the solutions to spaceflight optimization problems with bang-bang control. The rocket is subject to an inverse square gravitational force and to a thrust force with constant magnitude. The mass decreases linearly with time. It is assumed that an optimal impulsive solution has been obtained for a problem with given initial and final conditions. The method may then be used to obtain first-order corrections to the initial values of the costate variables.

Contents

The impulsive solution is defined to be the limit of the bounded thrust solutions as β, the fuel burning rate magnitude increases without bound. The thrust magnitude is $F=c\beta$ and c is the constant exhaust velocity. The corrections are the first-and higher-order terms of the Taylor series expansions of the variables about $\epsilon=0$, where $\epsilon=1/\beta$.

The problem of obtaining corrections to impulsive solutions has been studied independently by this author and several men[1-5] associated with Princeton University. References 1 and 3 are concerned with the problem of constant F/m, whereas Refs. 2 and 4 extend the work to the problem considered in this paper. The latter works make use of expansions in terms of two parameters; namely, initial thrust acceleration and the rocket jet exhaust velocity. Reference 5 considers applications to low-thrust mission analysis. The present paper is an extension of Ref. 6 to cover general initial conditions as well as final conditions.

The equations of motion are $\ddot{y}=(F/m)L(\lambda)+G(t,y)$ where $L(\lambda)$ is the optimal steering vector $\lambda/|\lambda|$. The costate vector λ is the solution to the costate equations $\ddot{\lambda}=Q(t,y,\lambda)$.

Let t_k and $\bar{t}_k$ be the initial and final times on the kth thrust arc for $k=1,2,...N$. Let κ be the so-called "switching function" satisfying the equation $\dot{\kappa}=(c/m)\ U(\lambda,\dot{\lambda})$ where $U=\lambda^T\dot{\lambda}/|\lambda|$. The necessary conditions of optimality include the conditon $\kappa=0$ at times t_k and $\bar{t}_k$ for $k=1,2,...,N$ except sometimes for time t_1 and time $\bar{t}_N$ if they correspond to the initial time t_0 and the final time t_f, respectively.

Assume that for each positive value of ϵ in some neighborhood of zero there is a solution y, $\dot{y}$, λ, $\dot{\lambda}$, κ, m, $t_k(\epsilon)$, $\bar{t}_k(\epsilon)$, $t_f(\epsilon)$ to the boundary-condition problem. Here y for example, is considered to be a function $y(t,\epsilon)$ of the two arguments, t and ϵ.

The multi-point boundary-condition problem for the impulsive case involves the choice of y_o, $\dot{y}_o$, λ_o, $\dot{\lambda}_0$, κ_0, t_1, t_2, ..., t_N, and t_f such that $\kappa(t_k,0)=0$ for $k=1,2,...,N$; such that $\dot{\kappa}(t_k,0)=0$ for $k=1,2,...,N$ (except sometimes for $k=1$ and/or $k=N$); and such that the given initial and final boundary conditions, including transversality conditions and a scaling condition upon λ, are satisfied.

Let $y_k(\epsilon)=y[t_k(\epsilon),\epsilon]$, $\bar{y}_k(\epsilon)=y[\bar{t}_k(\epsilon),\epsilon]$, $y_\epsilon=\partial y/\partial\epsilon$, $y_t=\dot{y}=\partial y/\partial t$, $y_{k\epsilon}=y_{\epsilon k}+t_{k\epsilon}\dot{y}_k$, etc. Also let $L^*(t,\epsilon)=L[\lambda(t,\epsilon)]$, $G^*(t,\epsilon)=G[t(\epsilon),y(t,\epsilon)]$, and so on.

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Suppose for example that one of the constraints is $y_f=a$. Then $y_f(\epsilon)\equiv a$ is an identity in ϵ. Therefore, $y_{f\epsilon}(\epsilon)\equiv 0$, $y_{f\epsilon\epsilon}(\epsilon)\equiv 0$, etc. If 4_f corresponds to $\bar{y}_N$, then $\bar{y}_{N\epsilon}\equiv 0$. This paper gives an expression which relates $\bar{y}_{k\epsilon}(0)$ to $y_{k\epsilon}(0)$. Moreover, $y_{k+1,\epsilon}$ may be written in terms of $\bar{y}_{k\epsilon}$, $\dot{\bar{y}}_{k\epsilon}$, $\bar{t}_{k\epsilon}$, and $t_{k+1,\epsilon}$. Ultimately the condition $y_{f\epsilon}(0)=0$ can be expressed as a linear equation in the unknowns $\lambda_{o\epsilon}$, $\lambda_{o\epsilon}$, $t_{1\epsilon}(0)$, etc. Similarly the other boundary conditions lead to linear equations. The linear equations may be solved for the unknown derivatives. Once $\lambda_{o\epsilon}(0)$, for example, has been calculated, one may correct the impulsive value $\lambda_o(0)$ by adding the first-order correction $\epsilon\lambda_{o\epsilon}(0)$ with $\epsilon=1/\beta$.

Let $\Delta t_k=\bar{t}_k-t_k$ and $\Delta m_k=\bar{m}_k-m_k$. Since $\Delta m_k=-\Delta t_k/\epsilon$, L'Hospital's rule gives the equation $\Delta t_k(0)=-\Delta m_k(0)$. In general

$$\frac{d^n\Delta m_k}{d\epsilon^n}=-\frac{1}{n+1}\frac{d^{n+1}\Delta t_k}{d\epsilon^{n+1}}$$

at $\epsilon=0$. Thus, for example, once $t_{k\epsilon}(0)$ is known, $\bar{t}_{k\epsilon}(0)$ can be easily calculated. Rather than considersing $t_{k\epsilon}(0)$ and $\bar{t}_{k\epsilon}(0)$ to be the unknowns on the k-th thrust arc, the unknowns will be $t_{k\epsilon}$ (or $\bar{t}_{k\epsilon}$) and $\Delta m_{k\epsilon}$.

Derivatives over coast arcs are easily obtained. For example

$$y_{k+1,\epsilon}=t_{k+1,\epsilon}y_{k+1}+[\partial y_{k+1}/\partial\bar{y}_k](\bar{y}_{k\epsilon}-\bar{t}_{k\epsilon}\dot{\bar{y}}_k)$$
$$+[\partial y_{k+1}/\partial\dot{\bar{y}}_k](\dot{\bar{y}}_{k\epsilon}-\bar{t}_{k\epsilon}\bar{G}_k^*)$$

The derivative of κ can be derived from the relationship $\kappa_{k+1}=\bar{\kappa}_{k\epsilon}+(c/\bar{m}_k)(|\lambda_{k+1}|-|\bar{\lambda}_k|)$. Thus

$$\kappa_{k+1,\epsilon}=\bar{\kappa}_{k\epsilon}+(c/\bar{m}_k)\{t_{k+1,\epsilon}U_{k+1}^*-t_{k\epsilon}\bar{U}_k^*$$
$$+(1/|\bar{\lambda}_k|)[\lambda_{k+1}^T(\lambda_{k+1,\epsilon}-t_{k+1,\epsilon}\dot{\lambda}_{k+1})-\bar{\lambda}_k^T(\bar{\lambda}_{k\epsilon}-t_{k\epsilon}\dot{\bar{\lambda}}_k)]\}$$

If $\bar{\kappa}_k\equiv 0$, $\kappa_{k+1}\equiv 0$, and $\epsilon=0$, the latter equation reduces to the simple condition

$$\lambda_{k+1}^T\lambda_{k+1,\epsilon}-\lambda_k^T\bar{\lambda}_{k\epsilon}=0 \tag{1}$$

Since $\ddot{y}=(c\beta/m)L+G$, integration by parts gives

$$\Delta y_k=\Delta t_k\dot{y}_k-c\epsilon\bar{m}_k\left(\ln\frac{m_k}{m_k}\right)L_k^*$$
$$-c\epsilon[mL^*]_{t_k}^{\bar{t}_k}+\int_{t_k}^{\bar{t}_k}\int_{t_k}^{\tau}G^*dtd\tau+\sigma(\epsilon^2)$$

$$\Delta\dot{y}_k=c\left(\ln\frac{m_k}{\bar{m}_k}\right)L_k^*+c\epsilon m_k\left(\ln\frac{m_k}{\bar{m}_k}\right)L_{t_k}^*$$
$$+c\epsilon[mL_t^*]_{t_k}^{\bar{t}_k}+\int_{t_k}^{\bar{t}_k}G^*dt+\sigma(\epsilon^3)$$

The next step is to find expressions for the integrals of G^*. Since $\dot{m}=-1/\epsilon$, repeated integration by parts gives

$$\int_{t_k}^{\bar{t}_k}G^*dt=-\epsilon[mG^*]_{t_k}^{\bar{t}_k}+\epsilon\int_{t_k}^{\bar{t}_k}mG_t^*dt$$
$$=-\epsilon[mG^*]_{t_k}^{\bar{t}_k}-\tfrac{1}{2}\epsilon^2[m^2G_t^*]_{t_k}^{\bar{t}_k}-\tfrac{1}{4}c\epsilon^2[m^2G_y^*L^*]_{t_k}^{\bar{t}_k}+\sigma(\epsilon^3)$$

Table 1 True, impulsive, and first-order solutions to intercept problem

	$\bar{t}_1$	λ_{11}	λ_{12}	λ_{13}	$\dot{\lambda}_{11}$	$\dot{\lambda}_{12}$	$\dot{\lambda}_{13}$
True solution	86.00	0.6618	0.3727	0.6505	−.001725	−.001198	−.001596
Impulsive	76.64	0.6631	0.3557	0.6587	−.001726	−.001149	−.001617
Corrected	84.04	0.6623	0.3726	0.6503	−.001726	−.001198	−.001596

and so on. The double integral is simply

$$\int_{t_k}^{\bar{t}_k}\int_{t_k}^{\tau} G^*\mathrm{d}t\mathrm{d}\tau = \tfrac{1}{2}\epsilon^2[m^2G^*]_{t_k}^{\bar{t}_k} + \epsilon\Delta t_k m_k G_k^* + \sigma(\epsilon^3)$$

Since $\ddot{\lambda} = Q(t,y,\lambda)$,

$$\Delta\lambda_k = \Delta t_k\dot{\lambda}_k + \sigma(\epsilon^2) \qquad \Delta\dot{\lambda}_k = -\epsilon[mQ^*]_{t_k}^{\bar{t}_k} + \sigma(\epsilon^2)$$

Since $\dot{\kappa} = (c/m)U(\lambda,\dot{\lambda})$,

$$\Delta\kappa_k = c\epsilon\ln(m_k/\bar{m}_k)(U_k^* + \epsilon U_{t_k}^* m_k) + \lambda(\epsilon^3)$$

Letting ϵ approach zero, one obtains $\Delta y_k(0) = 0$, $\Delta\dot{y}_k(0) = c\ln(m_k/\bar{m}_k)L_k^*$, $\Delta\lambda_k(0) = 0$, $\Delta\dot{\lambda}_k(0) = 0$, and $\Delta\kappa_k(0) = 0$. Since $|L_k^*| = 1$, it follows that at $\epsilon = 0$, $\Delta V_k = c\epsilon\ln(m_k/\bar{m}_k)$, $\bar{m}_k = m_k e^{-\Delta V_k/c}$, and $\Delta\dot{y}_k = \Delta V_k L_k^*$. Here $\Delta V_k = |\Delta\dot{y}_k|$.

Taking derivatives of the expressions obtained for Δy_k, etc., one obtains

$$\Delta y_{k\epsilon}(0) = -\Delta m_k\dot{y}_k - \bar{a}_k L_k^*$$

$$\Delta\dot{y}_{k\epsilon}(0) = \frac{c}{m_k\bar{m}_k}(\Delta m_k m_{k\epsilon} - m_k\Delta m_{k\epsilon})L_k^* + \Delta V_k L_{k\epsilon}^* - \Delta m_k G_k^* + a_k L_{tk}^*$$

$$\Delta\lambda_{k\epsilon}(0) = -\Delta m_k\dot{\lambda}_k,\ \Delta\dot{\lambda}_{k\epsilon}(0) = -\Delta m_k Q_k^*$$

$$\Delta\kappa_{k\epsilon}(0) = U_k^*\Delta V_k$$

$$\Delta\kappa_{k\epsilon\epsilon}(0) = \frac{2c}{m_k\bar{m}_k}U_k^*(\Delta m_k m_{k\epsilon} - m_k\Delta m_{k\epsilon}) + 2\Delta V_k U_{k\epsilon}^* + 2a_k U_{tk}^*$$

where $\bar{a}_k = \Delta V_k\bar{m}_k + \Delta m_k c$ and $a_k = \Delta V_k m_k + \Delta m_k c$.

If $\dot{\kappa} = 0$, then $U_k^* = 0$ so that $\Delta\kappa_{k\epsilon} = 0$. Then the condition $\kappa_{k\epsilon}(0) = 0$ implies $\bar{\kappa}_{k\epsilon} = 0$. In this case the conditon $\bar{\kappa}_{k\epsilon\epsilon} = 0$ should be employed rather than one of the conditions, $\bar{\kappa}_{k\epsilon}(0) = 0$ and $\kappa_{k\epsilon}(0) = 0$, in obtaining the first-order corrections. The condition $\bar{\kappa}_{k\epsilon\epsilon} = 0$ is simply

$$\dot{\lambda}_k^T\lambda_{k\epsilon} + \lambda_k^T\dot{\lambda}_{k\epsilon} = -\frac{a_k}{\Delta V_k}(\dot{\lambda}_k^T\dot{\lambda}_k + \lambda_k^T Q_k^*) \qquad (2)$$

For each intermediate thrust, conditions (1) and (2) apply and there are two unknowns; namely, $t_{k\epsilon}$ and $\Delta m_{k\epsilon}$ on the kth thrust arc.

As an example consider a thrust-coast intercept problem in which the vehicle moves from a given point in state space to a point with given position components. The time of intercept is also specified. In this problem t_0 must be identified with t_1. Since $\lambda_1^T\lambda_1 = 1$,

$$\lambda_1^T\lambda_{1\epsilon} = 0 \qquad (3)$$

Since y_2 is fixed, we have $y_{2\epsilon} = 0$ so that

$$\frac{\partial y_2}{\partial\bar{y}_1}\bar{y}_{\epsilon 1}^+ + \frac{\partial y_2}{\partial\dot{\bar{y}}_1}\dot{\bar{y}}_{\epsilon 1}^+ = 0$$

But

$$\bar{y}_{\epsilon 1}^+ = \bar{y}_{1\epsilon} - \bar{t}_{1\epsilon}\dot{\bar{y}}_1 = \Delta y_{1\epsilon} + \Delta m_1\dot{y}_1 = -a_1\lambda_1$$

and at $\epsilon = 0$,

$$\dot{\bar{y}}_{\epsilon 1}^+ = \dot{\bar{y}}_{1\epsilon} - \bar{t}_{1\epsilon}\ddot{\bar{y}}_1^+ = \Delta\dot{y}_{1\epsilon} + \Delta m_1 G_1^*$$

$$= -\frac{c}{\bar{m}_1}\Delta m_{1\epsilon}\lambda_1 + \Delta V_1\lambda_{1\epsilon} + a_1(I - \lambda_1\lambda_1^T)\dot{\lambda}_1$$

Therefore,

$$-a_1\frac{\partial y_2}{\partial\bar{y}_1}\lambda_1 + \frac{\partial y_2}{\partial\dot{\bar{y}}_1}\left[-\frac{c}{\bar{m}_1}\Delta m_{1\epsilon}\lambda_1 + \Delta V_1\lambda_{1\epsilon} + a_1(I - \lambda_1\lambda_1^T)\dot{\lambda}_1\right] = 0 \qquad (4)$$

Equations (3) and (4) may be written as

$$\begin{bmatrix} -\dfrac{c}{\bar{m}_1}\dfrac{\partial y_2}{\partial\dot{\bar{y}}_1}\lambda_1 & \Delta V_1\dfrac{\partial y_2}{\partial\dot{\bar{y}}_1} \\ 0 & \lambda_1^T \end{bmatrix}\begin{bmatrix}\Delta m_{1\epsilon} \\ \lambda_{1\epsilon}\end{bmatrix} = \begin{bmatrix}\gamma \\ 0\end{bmatrix}$$

where $\gamma = a_1(\partial y_2/\partial\bar{y}_1)\lambda_1 - a_1(\partial y_2/\partial\dot{\bar{y}}_1)(I - \lambda_1\lambda_1^T)\dot{\lambda}$. Therefore

$$\begin{bmatrix}\Delta m_{1\epsilon} \\ \lambda_{1\epsilon}\end{bmatrix} = \begin{bmatrix} -\dfrac{\bar{m}_1}{c}\lambda_1^T\left(\dfrac{\partial y_2}{\partial\dot{\bar{y}}_1}\right)^{-1} & \dfrac{\bar{m}_1}{c}\Delta V_1 \\ \dfrac{1}{\Delta V_1}(I - \lambda_1\lambda_1^T)\left(\dfrac{\partial y_2}{\partial\dot{\bar{y}}_1}\right)^{-1} & \lambda_1 \end{bmatrix}\begin{bmatrix}\gamma \\ 0\end{bmatrix} \qquad (5)$$

We observe that $\lambda_2 = (\partial y_2/\partial y_1)\lambda_1 + (\partial y_2/\partial\dot{y}_1)\dot{\lambda}_1$. In an intercept problem the transversality conditions imply that $\lambda_2 = 0$. Therefore $\dot{\lambda}_1 = -(\partial y_2/\partial\dot{y}_1)^{-1}(\partial y_2/\partial y_1)\lambda_1$. Utilizing the latter equation and Eq. (5), it may be shown that $\Delta m_{1\epsilon} = (\bar{m}_1 a_1/c)\lambda_1^T\dot{\lambda}_1$ and $\lambda_{1\epsilon} = (2a_1/\Delta V_1)[(\lambda_1^T\dot{\lambda}_1)\lambda_1 - \dot{\lambda}_1]$ at $\epsilon = 0$. Since $\lambda_2 = 0$, it may also be shown that

$$\dot{\lambda}_{1\epsilon} = \left(\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right)^{-1}\left(a_1\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\lambda_1 - \frac{\partial\lambda_2}{\partial\bar{y}_1}\Delta y_{\epsilon 1} - \frac{\partial y_2}{\partial\bar{y}_1}\lambda_{1\epsilon}\right)$$

Let $t_1 = 0$, $t_2 = 380$, $\mu = 0.388\times10^{15}$ m^3/sec^2, $c = 4100$ m/sec, $\beta = 22$ kg-sec/m, $m_1 = 0.168920\times10^5$ kg-sec^2/m, $y_1^T = (0.287253\times10^7\text{m},\ 0.590785\times10^7\text{m},\ 0.777376\times10^5\text{m})$, $\dot{y}_1^T = (-7326.35\ \text{m/sec},\ 3219.14\ \text{m/sec},\ -474.472\ \text{m/sec})$, $y_2^T = (0,\ 0.655630\times10^7\text{m},\ 0)$. Table 1 summarizes the results of an impulsive solution and first-order correction.

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